

# WEIGHTED BERGMAN SPACES OF DOMAINS WITH LEVI-FLAT BOUNDARY: GEODESIC SEGMENTS ON COMPACT RIEMANN SURFACES

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**ABSTRACT.** The aim of this study is to understand to what extent a 1-convex domain with Levi-flat boundary is capable of holomorphic functions with slow growth. This paper discusses the case of the space of all the geodesic segments on a hyperbolic compact Riemann surface, which is a typical example of such a domain in the sense that its realization as a holomorphic disk bundle has the best possible Diederich–Fornaess index  $1/2$ . Our main finding is an integral formula that produces holomorphic functions on the domain from holomorphic differentials on the base Riemann surface via optimal  $L^2$ -jet extension, and, in particular, it is shown that the weighted Bergman spaces of the domain is infinite dimensional for all the order greater than  $-1$  beyond  $-1/2$ , the limiting order until which known  $L^2$ -estimates for the  $\bar{\partial}$ -equation work. Some applications are also given thanks to the generalized hypergeometric function  ${}_3F_2$  expressing the norm of the optimal  $L^2$ -jet extension: a proof for the Liouvilieness which does not appeal to the ergodicity of the Levi foliation, and a Forelli–Rudin construction for the disk bundle over the Riemann surface.

## 1. INTRODUCTION

Denote by  $\mathbb{D}$  the unit disk in  $\mathbb{C}$ , and let  $\Sigma = \mathbb{D}/\Gamma$  be a compact Riemann surface of genus  $\geq 2$  where  $\Gamma$  is a Fuchsian group  $< \text{Aut}(\mathbb{D}) = PSU(1, 1)$ . We consider a quotient of the bidisk by  $\Gamma$ ,  $\Omega := \mathbb{D} \times \mathbb{D}/\Gamma$ , where  $\Gamma$  acts on  $\mathbb{D} \times \mathbb{D}$  diagonally, namely,

$$\gamma \cdot (z, w) = (\gamma z, \gamma w)$$

for each  $\gamma \in \Gamma$ ,  $(z, w) \in \mathbb{D} \times \mathbb{D}$ . The space  $\Omega$  is, geometrically speaking, identified with the space of all the geodesic segments on  $\Sigma$  with respect to the Poincaré metric: a point  $[(z, w)] \in \mathbb{D} \times \mathbb{D}/\Gamma$  is identified with the geodesic on  $\Sigma$  obtained by projecting down the geodesic  $\overline{zw}$  on  $\mathbb{D}$  connecting  $z$  and  $w$ . The purpose of this paper is to describe  $\mathcal{O}(\Omega)$ , the space of holomorphic functions on  $\Omega$ , in other words, that of  $\Gamma$ -invariant holomorphic functions  $\mathcal{O}(\mathbb{D} \times \mathbb{D})^\Gamma$  on  $\mathbb{D} \times \mathbb{D}$ , in an explicit way.

We may regard  $\Omega$  as a locally-trivial holomorphic  $\mathbb{D}$ -bundle over  $\Sigma$  by its first projection  $\pi|_\Omega: \Omega \rightarrow \Sigma$ . Then,  $\Omega$  is naturally realized as a relatively compact domain in a holomorphic  $\mathbb{CP}^1$ -bundle over  $\Sigma$  denoted by  $X := \mathbb{D} \times \mathbb{CP}^1/\Gamma$  where  $\Gamma$  acts on  $\mathbb{D} \times \mathbb{CP}^1$  diagonally again. The boundary of  $\Omega$  in this realization is known to be real-analytic Levi-flat, whose Levi foliation agrees with weakly stable foliation of the geodesic flow on the hyperbolic surface  $\Sigma$ , and regarded as a standard example

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of Levi-flat real hypersurface with positive normal bundle (Brunella [5]), whereas not much examples are known about.

The holomorphic convexity of  $\Omega$  was first studied by Diederich and Ohsawa [9]. They showed that  $\Omega$  is 1-convex and its maximal compact analytic set  $D$  is the quotient of the diagonal set  $\Delta \subset \mathbb{D} \times \mathbb{D}$ , which is isomorphic to  $\Sigma$ , by observing that

$$-\log \left( 1 - \left| \frac{w - z}{1 - \bar{z}w} \right| \right)$$

is a plurisubharmonic exhaustion on  $\Omega$  which is strictly plurisubharmonic on  $\Omega \setminus \Delta$ . We know therefore that  $\Omega$  possesses a plenty of holomorphic functions. However, we have never seen an explicit construction of non-trivial holomorphic functions on  $\Omega$  except for the functions given by a Poincaré series

$$\sum_{\gamma \in \Gamma} (\gamma(z) - \gamma(w))^N$$

where  $N \geq 2$ , which was observed by Ohsawa [21].

The goal of this paper is to give a concrete description of  $\mathcal{O}(\Omega)$ , and show that  $\Omega$  actually has a plenty of holomorphic functions with slow growth, namely, those belonging to the weighted Bergman space  $A_\alpha^2(\Omega)$  of order  $\alpha > -1$  (for its definition, see §2.2).

**Main Theorem.** *We have an injective linear map*

$$I: \bigoplus_{N=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes N}) \hookrightarrow \bigcap_{\alpha > -1} A_\alpha^2(\Omega) \subset \mathcal{O}(\Omega)$$

having dense image in  $\mathcal{O}(\Omega)$  equipped with compact open topology and expressed by

$$I(\psi)(z, w) = \begin{cases} \frac{1}{B(N, N)} \int_{\tau \in \mathbb{D}} \frac{\psi(\tau)(d\tau)^{\otimes N}}{[w, \tau, z]^{\otimes(N-1)}} & \text{for } N \geq 1, \\ \text{the constant } \psi & \text{for } N = 0 \end{cases}$$

as a function in  $\mathcal{O}(\mathbb{D} \times \mathbb{D})^\Gamma$  for  $\psi = \psi(\tau)(d\tau)^{\otimes N} \in H^0(\Sigma, K_\Sigma^{\otimes N}) \subset H^0(\mathbb{D}, K_\mathbb{D}^{\otimes N})$  where  $\tau$  is the coordinate of  $\mathbb{D}$ , the universal cover of  $\Sigma$ . Here we denoted

$$[w, \tau, z] := \frac{(w - z)d\tau}{(w - \tau)(\tau - z)},$$

an  $\text{Aut}(\mathbb{D})$ -invariant meromorphic 1-form in  $\tau$  on  $\mathbb{D}$ , and  $B(p, q)$  is the beta function.

Note that the canonical ring  $R(\Sigma) := \bigoplus_{N=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes N})$  is identified with the graded ring  $\bigoplus_{N=0}^{\infty} H^0(D, \mathcal{I}_D^N / \mathcal{I}_D^{N+1})$  associated with the filtered ring of jets of holomorphic functions along  $D$ , where  $\mathcal{I}_D$  denotes the ideal sheaf of  $D$ . The proof of Main Theorem is carried out by looking for the extension of jets of holomorphic functions belonging to  $H^0(D, \mathcal{I}_D^N / \mathcal{I}_D^{N+1})$  to holomorphic functions on  $\Omega$  with minimal  $L^2$  norm, and our finding is that the operator  $I$  given by the formula above is the optimal  $L^2$ -jet extension operator. We hope that this example would give some insight to pursue the  $L^2$ -jet extension theorem with optimal constant (cf. [23], [16], [8]).

The virtue of our Main Theorem can be seen in the following Corollary.

**Corollary 1.** *The weighted Bergman space  $A_\alpha^2(\Omega)$  is infinite dimensional for  $\alpha > -1$ .*

The existence of non-constant holomorphic function with such a slow growth seems to be unreachable by known  $L^2$ -estimates for the  $\bar{\partial}$ -equation, even by the  $L^2$  extension theorem of Ohsawa–Takegoshi type with optimal constant established by Błocki [4] and Guan–Zhou [14] (see in particular [20] and [14, Theorem 3.6]). First note that the existence of such a holomorphic function (without norm estimate) is obvious for domains with Stein neighborhoods, for which even we have a plenty of bounded holomorphic functions. Our domain  $\Omega$ , however, does not have a Stein neighborhood since its boundary is compact Levi-flat, nor admit a bounded holomorphic functions except constant functions (see §7.1). Modifying the argument in [14, Theorem 3.6] with a plurisubharmonic exhaustion of self bounded gradient on  $\Omega$  (cf. [3], [18], [15], [6], [22]), we may construct holomorphic functions on  $\Omega$  with  $L^2$  estimates by extending weighted  $L^2$  holomorphic functions on a fiber of  $\pi|_\Omega: \Omega \rightarrow \Sigma$ , but it barely gives us holomorphic functions of  $A_\alpha^2(\Omega)$  for  $\alpha > -1/2$ . This  $1/2$  comes from the Diederich–Fornaess index of our domain  $\Omega \subset X$  (cf. [1]), and is the best possible value according to Fu–Shaw [12], and Brinkschulte and the author [2]. This situation reveals that there is, still, some room to improve our understanding of  $L^2$ -estimates for the  $\bar{\partial}$ -equation, in particular, the twisting technique.

In this context, we would like to emphasize a work of Chen [7], which motivated this work. He studied the weighted Bergman spaces of pseudoconvex domains in Euclidean spaces  $\mathbb{C}^n$  with  $C^2$ -smooth boundary, and proved a Hörmander type  $L^2$ -estimate for the  $\bar{\partial}$ -equation in weighted  $L^2$  spaces of any order  $> -1$ , whatever their Diederich–Fornaess index are, although there he could exploit the background strictly plurisubharmonic exhaustion, the Kohn weight.

As an application of our description of  $\mathcal{O}(\Omega)$ , the following classical fact dated back to Hopf [17] (cf. [25], [24], [13], [10]) is reproved without appealing to the ergodicity of the Levi foliation on the boundary, in fact, even without looking at  $\partial\Omega$ .

**Corollary 2.** *The Hardy space  $A_{-1}^2(\Omega)$  consists only of constant functions. In particular, there is no bounded holomorphic function on  $\Omega$  except constant functions.*

As an another application, we shall give a Forelli–Rudin construction (cf. [11], [19]) for weighted Bergman kernels of  $\Omega$ .

**Corollary 3.** *For  $\alpha > -1$ , the weighted Bergman kernel  $B_\alpha((z, w); (z', w'))$  of  $A_\alpha^2(\Omega)$  (see §2.2 for the choice of the measure and weight) has the following expression*

$$\begin{aligned} & B_\alpha((z, w); (z', w')) \\ &= \frac{\Gamma(\alpha + 2)}{\pi^2(4g - 4)} + \frac{1}{\pi} \sum_{N=1}^{\infty} \frac{1}{c_{N,\alpha}} \frac{1}{B(N, N)^2} \int_{\tau \in \overline{z\bar{w}}} \int_{\tau' \in \overline{z'\bar{w}'}} \frac{B_N(\tau, \tau')(d\tau \otimes \overline{d\tau'})^{\otimes N}}{([w, \tau, z] \otimes [\overline{w'}, \overline{\tau'}, \overline{z'}])^{\otimes(N-1)}} \end{aligned}$$

where  $g$  is the genus of  $\Sigma$ ,  $B_N(\tau, \tau')(d\tau \otimes \overline{d\tau'})^{\otimes N}$  is the Bergman kernel of  $K_\Sigma^{\otimes N}$ , and

$$c_{N,\alpha} := \frac{\Gamma(N + 1)}{\Gamma(N + 2 + \alpha)} {}_3F_2 \left( \begin{matrix} N + 1, N, N \\ 2N, N + 2 + \alpha \end{matrix}; 1 \right).$$

Here  $\Gamma(z)$  and  ${}_3F_2(\cdots; z)$  denote the gamma function and the generalized hypergeometric function respectively.

At the end, we shall explain the invariant holomorphic functions constructed by Ohsawa [21] from our viewpoint.

**Corollary 4.** *For  $N \geq 2$ , we have*

$$I\left(\sum_{\gamma \in \Gamma} \gamma^* d\tau^{\otimes N}\right)(z, w) = \sum_{\gamma \in \Gamma} (\gamma(z) - \gamma(w))^N$$

where  $\sum_{\gamma \in \Gamma} \gamma^* d\tau^{\otimes N} \in H^0(\mathbb{D}, K_{\mathbb{D}}^{\otimes N})$  is regarded as a holomorphic  $N$ -differential on  $\Sigma$  à la Poincaré.

It might be of interest that the extension operator  $I$  enjoys such an algebraic identity although it is not a ring homomorphism from  $R(\Sigma)$  to  $\mathcal{O}(\Omega)$ .

This paper is organized as follows: In §2, we explain background materials to set up our notation. In §3, we consider the Taylor expansion of  $f \in \mathcal{O}(\Omega)$  along the maximal compact analytic set  $D$ , and derive a system of the  $\bar{\partial}$ -equations on  $\Sigma$  which the Taylor coefficients of  $f$  satisfy. In §4, we examine the  $L^2$ -minimal solution to the system of the  $\bar{\partial}$ -equation. We are able to compute the  $L^2$ -norm of the solution thanks to the spectral decomposition of the  $\bar{\partial}$ -Laplacian acting on the canonical ring of  $\Sigma$ . In §5, we consider the formal holomorphic function obtained by the  $L^2$ -minimal solution to the system of the  $\bar{\partial}$ -equation, and see that the formal solution converges and defines a genuine holomorphic function. Then, in §6, we complete the proof of Main Theorem. In particular, we show the explicit expression of the extension operator  $I$ . §7 contains the proof for Corollary 2–4, some applications of our argument.

## 2. PRELIMINARIES

**2.1.  $\bar{\partial}$ -equation and Green operator on the canonical ring of  $\Sigma$ .** We give a quick review for  $\bar{\partial}$ -formalism on compact Riemann surfaces to explain our notation and convention.

Let  $\Sigma$  be a compact Riemann surface,  $g$  a Kähler metric of  $\Sigma$  with the fundamental form  $\omega_g$ , and  $(L, h) \rightarrow \Sigma$  a hermitian holomorphic line bundle. We consider  $L^2$  Dolbeault complex

$$L_{(2)}^{(0,0)}(\Sigma, L) \xrightarrow{\bar{\partial}} L_{(2)}^{(0,1)}(\Sigma, L)$$

by completing smooth Dolbeault complex

$$C^{(0,0)}(\Sigma, L) \xrightarrow{\bar{\partial}} C^{(0,1)}(\Sigma, L)$$

with inner product

$$\langle \langle u, v \rangle \rangle = \int_{\Sigma} \langle u, v \rangle_{h,g} \omega_g$$

and the unique closed extension of  $\bar{\partial}$ . It is well known that their cohomology groups are isomorphic  $H_{(2)}^*(\Sigma, L) \simeq H^*(\Sigma, L)$ .

Using local frame  $e_{\alpha}$  of  $L$ , and local coordinate  $z_{\alpha}$  of  $\Sigma$ , we write locally

$$u = u_{\alpha} e_{\alpha}, \quad |u|_h^2 = h_{\alpha} |u_{\alpha}|^2, \quad v = v_{\alpha} e_{\alpha} \otimes d\bar{z}_{\alpha}, \quad \omega_g = g_{\alpha} i dz_{\alpha} \wedge d\bar{z}_{\alpha}, \quad |v|_{h,g}^2 = \frac{h_{\alpha} |v_{\alpha}|^2}{g_{\alpha}}$$

for  $u \in L_{(2)}^{(0,0)}(\Sigma, L)$  and  $v \in L_{(2)}^{(0,1)}(\Sigma, L)$ . Then, the adjoint operator  $\bar{\partial}_L^*$  of  $\bar{\partial}$ , and the  $\bar{\partial}$ -Laplacians  $\square_L^{(0)} = \bar{\partial}_L^* \bar{\partial}$  and  $\square_L^{(1)} = \bar{\partial} \bar{\partial}_L^*$ , which are essentially self-adjoint, are expressed as

$$\begin{aligned}\bar{\partial}_L^* v &= \frac{-1}{g_\alpha} \left( \frac{\partial v_\alpha}{\partial z_\alpha} + \frac{\partial \log h_\alpha}{\partial z_\alpha} v_\alpha \right) e_\alpha = \frac{-1}{g_\alpha h_\alpha} \frac{\partial(h_\alpha v_\alpha)}{\partial z_\alpha} e_\alpha, \\ \square_L^{(0)} u &= \frac{-1}{g_\alpha} \left( \frac{\partial^2 u_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha} + \frac{\partial \log h_\alpha}{\partial z_\alpha} \frac{\partial u_\alpha}{\partial \bar{z}_\alpha} \right) e_\alpha,\end{aligned}$$

and

$$\square_L^{(1)} v = \frac{\partial}{\partial \bar{z}_\alpha} \left( \frac{-1}{g_\alpha} \left( \frac{\partial v_\alpha}{\partial z_\alpha} + \frac{\partial \log h_\alpha}{\partial z_\alpha} v_\alpha \right) \right) e_\alpha \otimes d\bar{z}_\alpha$$

for  $u \in C^{(0,0)}(\Sigma, L)$  and  $v \in C^{(0,1)}(\Sigma, L)$ .

We denote by  $G_L^{(1)}$  the *Green operator* for  $L$ -valued  $(0,1)$ -forms, that is,  $G_L^{(1)}$  is a bounded linear operator on  $L_{(2)}^{(0,1)}(\Sigma, L)$  preserving  $C^{(0,1)}(\Sigma, L)$ , and satisfying

$$\square_L^{(1)} G_L^{(1)} = G_L^{(1)} \square_L^{(1)} = I - H_L^{(1)}$$

where  $I$  is the identity map and  $H_L^{(1)} : L_{(2)}^{(0,1)}(\Sigma, L) \rightarrow \text{Ker } \square_L^{(1)}$  is the orthogonal projection to  $\bar{\partial}$ -harmonic forms. In particular, when the Dolbeault cohomology group  $H^1(\Sigma, L) \simeq \text{Ker } \square_L^{(1)}$  vanishes,  $\square_L^{(1)}$  is invertible linear operator and  $G_L^{(1)} = \left( \square_L^{(1)} \right)^{-1}$ , hence, it follows that for any  $v \in L_{(2)}^{(0,1)}(\Sigma, L)$ ,  $u := \bar{\partial}_L^* G_L^{(1)} v \in L_{(2)}^{(0,0)}(\Sigma, L)$  solves the  $\bar{\partial}$ -equation,  $\bar{\partial} u = v$ , and this  $u$  has the smallest  $L^2$  norm among all the solutions.

Now we assume that the genus of  $\Sigma$  is  $\geq 2$  in the rest of this paper, and fix its uniformization  $\Sigma = \mathbb{D}/\Gamma$  where  $\Gamma$  is a Fuchsian group. We refer the natural coordinate  $z$  of  $\mathbb{D} \subset \mathbb{C}$  as the *uniformizing coordinate*. We equip  $\Sigma$  with the *Poincaré metric*  $g$  whose fundamental form is

$$\omega_g = \frac{2i dz \wedge d\bar{z}}{(1 - |z|^2)^2}$$

on the uniformizing coordinate  $z$ . By abuse of notation, we denote the coefficient by

$$g(z) := \frac{2}{(1 - |z|^2)^2}.$$

We will work on  $L = K_\Sigma^{\otimes n}$  where  $K_\Sigma$  denotes the canonical bundle of  $\Sigma$ . The sections of  $K_\Sigma^{\otimes n}$  will be referred to as  $n$ -differentials, or  $(n,0)$ -differentials later. We also call  $K_\Sigma^{\otimes n}$ -valued  $(0,1)$ -forms  $(n,1)$ -differentials. The Poincaré metric induces a hermitian metric on  $K_\Sigma^{\otimes n}$ . In the uniformizing coordinate  $z$ , our normalization is

$$|(dz)^{\otimes n}|_g^2 = \left( \frac{1 - |z|^2}{\sqrt{2}} \right)^{2n} = g^{-n}.$$

The operators with respect to this metric will be denoted by  $\bar{\partial}_{K_\Sigma^{\otimes n}}^* =: \bar{\partial}_n^*$ ,  $\square_{K_\Sigma^{\otimes n}}^{(0)} =: \square_n^{(0)}$ ,  $\square_{K_\Sigma^{\otimes n}}^{(1)} =: \square_n^{(1)}$  and  $G_{K_\Sigma^{\otimes n}}^{(1)} =: G_n^{(1)}$  for short.

Classical facts show the vanishing of  $H^1(\Sigma, K_\Sigma^{\otimes n})$  for  $n \geq 1$ , hence, we have the following.

**Lemma 2.1.** *Let  $n \geq 1$  and  $v \in C^{(0,1)}(\Sigma, K_\Sigma^{\otimes n})$ . Then,  $u := \bar{\partial}_n^* G_n^{(1)} v \in C^{(0,0)}(\Sigma, K_\Sigma^{\otimes n})$  is the  $L^2$ -minimal solution to  $\bar{\partial}u = v$ .*

**2.2. Weighted  $L^2$ -norms and Bergman spaces.** We denote the coordinate of the bidisk  $\mathbb{D} \times \mathbb{D}$  in  $\mathbb{C}^2$  by  $(z, w)$ . Throughout this paper, we will also use another non-holomorphic coordinate  $(z, t)$  given by

$$\mathbb{D} \times \mathbb{D} \ni (z, w) \mapsto (z, t = \frac{w - z}{1 - \bar{z}w}) \in \mathbb{D} \times \mathbb{D},$$

whose inverse transformation is given by

$$\mathbb{D} \times \mathbb{D} \ni (z, t) \mapsto (z, w = \frac{t + z}{1 + \bar{z}t}) \in \mathbb{D} \times \mathbb{D}.$$

Note that this coordinate change identifies the diagonal  $\Delta \subset \mathbb{D} \times \mathbb{D}$  with a horizontal disk  $\mathbb{D} \times \{0\}$ .

Let  $\Sigma = \mathbb{D}/\Gamma$  be a compact Riemann surface of genus  $\geq 2$  as before, and consider the quotient space  $\Omega := \mathbb{D} \times \mathbb{D}/\Gamma$ , where  $\Gamma$  acts on  $\mathbb{D} \times \mathbb{D}$  diagonally. Let  $X := \mathbb{D} \times \mathbb{CP}^1/\Gamma$  where  $\Gamma$  acts on  $\mathbb{D} \times \mathbb{CP}^1$  diagonally again. The first projection on  $\mathbb{D} \times \mathbb{CP}^1$  induces a holomorphic submersion  $\pi: X \rightarrow \Sigma$ , which is a holomorphic  $\mathbb{CP}^1$  bundle, and  $\pi|\Omega$  is a holomorphic  $\mathbb{D}$ -bundle. Note that the quotient of the diagonal set  $D := \Delta/\Gamma$  is the maximal compact analytic set in  $\Omega$  biholomorphic to  $\Sigma$ .

We shall use a hermitian metric  $G$  on  $\Omega$  whose fundamental form is expressed as

$$\omega_G := \frac{2idz \wedge d\bar{z}}{(1 - |z|^2)^2} + \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4} \frac{i}{2} dw \wedge d\bar{w}.$$

Note that  $\omega_G$  agrees with  $idt \wedge d\bar{t}/2$  on each fiber of  $\pi|\Omega$ . We measure the  $L^2$ -norm of measurable functions on  $\Omega$  with respect to its volume form

$$dV := \frac{1}{2!} (\omega_G)^2 = \frac{4}{|1 - \bar{z}w|^4} \frac{i}{2} dz \wedge d\bar{z} \wedge \frac{i}{2} dw \wedge d\bar{w}$$

coming from  $G$  and the weight function of the form  $\delta^\alpha$  where

$$\delta := 1 - |t|^2 = 1 - \left| \frac{w - z}{1 - \bar{z}w} \right|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

The invariance of the metric and the weight function under the action of  $\Gamma$  follows from a direct computation. Note also that  $\Omega$  has finite volume with respect to  $dV$ .

Now we let

$$\begin{aligned} \langle \langle f, g \rangle \rangle_\alpha^2 &:= \frac{1}{\Gamma(\alpha + 1)} \int_\Omega f \bar{g} \delta^\alpha dV \\ &= \frac{1}{\Gamma(\alpha + 1)} \int_\Omega f(z, w) \overline{g(z, w)} \frac{4(1 - |z|^2)^\alpha (1 - |w|^2)^\alpha}{|1 - \bar{z}w|^{4+2\alpha}} \frac{i}{2} dz \wedge d\bar{z} \wedge \frac{i}{2} dw \wedge d\bar{w} \end{aligned}$$

for a measurable function  $f, g$  on  $\Omega$  and  $\alpha > -1$ , and define the *weighted  $L^2$  space* by

$$L_\alpha^2(\Omega) := \{f : \text{measurable } \mathbb{C}\text{-valued function on } \Omega \mid \|f\|_\alpha^2 := \langle \langle f, f \rangle \rangle_\alpha < \infty\}$$

and the *weighted Bergman space* by

$$A_\alpha^2(\Omega) := L_\alpha^2(\Omega) \cap \mathcal{O}(\Omega).$$

It is well known that the weighted  $L^2$  space is a separable Hilbert space, and the weighted Bergman space is its closed subspace.

We will also use weighted  $L^2$  Dolbeault complex on  $\Omega$  later:

$$L_{\alpha}^2(\Omega) = L_{(2),\alpha}^{(0,0)}(\Omega) \xrightarrow{\bar{\partial}} L_{(2),\alpha}^{(0,1)}(\Omega) \xrightarrow{\bar{\partial}} L_{(2),\alpha}^{(0,2)}(\Omega),$$

which is the completion of smooth Dolbeault complex

$$C^{(0,0)}(\Omega) \xrightarrow{\bar{\partial}} C^{(0,1)}(\Omega) \xrightarrow{\bar{\partial}} C^{(0,2)}(\Omega)$$

with inner product

$$\langle\langle u, v \rangle\rangle_{\alpha} := \int_{\Omega} \langle u, v \rangle_G \delta^{\alpha} dV$$

and the maximal closed extension of  $\bar{\partial}$ .

### 3. A SYSTEM OF $\bar{\partial}$ -EQUATIONS FOR DIFFERENTIALS ON $\Sigma$

In this section, we consider the Taylor expansion of  $f \in \mathcal{O}(\Omega)$  along the maximal compact analytic set  $D$ , and derive a system of the  $\bar{\partial}$ -equations on  $\Sigma$  which the Taylor coefficients satisfy.

**3.1.  $\bar{\partial}$ -equations for Taylor coefficients along the diagonal.** Let  $f = f(z, w) \in \mathcal{O}(\mathbb{D} \times \mathbb{D})$ . We shall derive a system of  $\bar{\partial}$ -equations which the Taylor coefficients of  $f$  along the diagonal  $\Delta \subset \mathbb{D} \times \mathbb{D}$  satisfies.

We shall work on the coordinate  $(z, t)$ , and put  $\tilde{f}(z, t) := f(z, w(z, t))$ . Although  $\tilde{f}$  is not holomorphic in  $z$  since so is the coordinate change, it satisfies instead

$$\begin{aligned} 0 &= \frac{\partial}{\partial \bar{z}} f(z, w) = \frac{\partial}{\partial \bar{z}} f(z, w(z, t(z, w))) \\ &= \frac{\partial}{\partial \bar{z}} \tilde{f}(z, t(z, w)) = \frac{\partial}{\partial \bar{z}} \tilde{f}\left(z, \frac{w - z}{1 - \bar{z}w}\right) \\ &= \frac{\partial \tilde{f}}{\partial \bar{z}} + \frac{\partial \tilde{f}}{\partial t} \frac{w(w - z)}{(1 - \bar{z}w)^2} \\ &= \frac{\partial \tilde{f}}{\partial \bar{z}} + \frac{t(t + z)}{1 - |z|^2} \frac{\partial \tilde{f}}{\partial t}. \end{aligned}$$

Let us denote the Taylor coefficients of  $f$  computed in the coordinate  $(z, t)$  by

$$f_n(z) := \frac{1}{n!} \frac{\partial^n \tilde{f}}{\partial t^n}(z, 0).$$

Then, they enjoy

$$\begin{aligned} 0 &= \frac{1}{n!} \frac{\partial^n}{\partial t^n} \left( \frac{\partial \tilde{f}}{\partial \bar{z}} + \frac{t(t + z)}{1 - |z|^2} \frac{\partial \tilde{f}}{\partial t} \right) \Big|_{t=0} \\ &= \frac{\partial f_n}{\partial \bar{z}} + \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \frac{t + z}{1 - |z|^2} \frac{\partial \tilde{f}}{\partial t} \Big|_{t=0} \\ &= \frac{\partial f_n}{\partial \bar{z}} + \frac{nz}{1 - |z|^2} f_n + \frac{n-1}{1 - |z|^2} f_{n-1} \end{aligned}$$

for  $n \geq 1$ .

3.2.  **$\bar{\partial}$ -equations on  $\Sigma$ .** Now we assume our  $f \in \mathcal{O}(\mathbb{D} \times \mathbb{D})$  is invariant under the action of  $\Gamma$ . Then, for each  $\gamma = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in \Gamma < PSU(1, 1)$ ,  $|\alpha|^2 - |\beta|^2 = 1$ ,

$$\begin{aligned} \tilde{f}(z, t) &= f(z, w(z, t)) = f(\gamma z, \gamma w(z, t)) \\ &= \tilde{f}(\gamma z, t(\gamma z, \gamma w(z, t))) = \tilde{f}(\gamma z, \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} t) \end{aligned}$$

holds and the Taylor coefficients satisfy

$$\begin{aligned} f_n(z) &= \frac{1}{n!} \frac{\partial^n}{\partial t^n} \tilde{f}(z, t) \Big|_{t=0} = \frac{1}{n!} \frac{\partial^n}{\partial t^n} \tilde{f}(\gamma z, \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} t) \Big|_{t=0} \\ &= \frac{1}{n!} \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} \frac{\partial^{n-1}}{\partial t^{n-1}} \frac{\partial \tilde{f}}{\partial t}(\gamma z, \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} t) \Big|_{t=0} \\ &= \frac{1}{n!} \left( \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} \right)^n \frac{\partial^n \tilde{f}}{\partial t^n}(\gamma z, \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} t) \Big|_{t=0} \\ &= \left( \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} \right)^n f_n(\gamma z). \end{aligned}$$

Since

$$\gamma^* dz = \frac{dz}{(\bar{\beta} z + \bar{\alpha})^2}, \quad \gamma^* d\bar{z} = \frac{d\bar{z}}{(\beta \bar{z} + \alpha)^2},$$

we have the invariance for

$$\begin{aligned} \gamma^* \left( f_n(z) \left( \frac{\sqrt{2} dz}{1 - |z|^2} \right)^{\otimes n} \right) &= f_n(\gamma z) \left( \frac{\sqrt{2} \gamma^* dz}{1 - |\gamma z|^2} \right)^{\otimes n} \\ &= \left( \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \bar{\alpha}} \right)^n \frac{1}{(\bar{\beta} z + \bar{\alpha})^{2n}} \frac{1}{(1 - |\gamma z|^2)^n} f_n(z) (\sqrt{2} dz)^{\otimes n} \\ &= \left( \frac{1}{|\beta \bar{z} + \alpha|^2} \right)^n \left( \frac{|\bar{\beta} z + \bar{\alpha}|^2}{|\bar{\beta} z + \bar{\alpha}|^2 - |\alpha z + \beta|^2} \right)^n f_n(z) (\sqrt{2} dz)^{\otimes n} \\ &= f_n(z) \left( \frac{\sqrt{2} dz}{1 - |z|^2} \right)^{\otimes n}. \end{aligned}$$

Now let us define

$$\varphi_n := f_n(z) \left( \frac{\sqrt{2} dz}{1 - |z|^2} \right)^{\otimes n},$$



which we shall call the  $n$ -th *associated differential* of  $f$ . Then,  $\varphi_n$  is a  $(n, 0)$ -differential on  $\Sigma$ , and satisfies  $\bar{\partial}\varphi_0 = 0$  and

$$\begin{aligned}\bar{\partial}\varphi_n &= \frac{\partial}{\partial\bar{z}} \left( f_n(z) \left( \frac{\sqrt{2}dz}{1-|z|^2} \right)^{\otimes n} \right) \otimes d\bar{z} \\ &= \left( \frac{\partial f_n}{\partial\bar{z}} + \frac{nzf_n}{1-|z|^2} \right) \left( \frac{\sqrt{2}dz}{1-|z|^2} \right)^{\otimes n} \otimes d\bar{z} \\ &= -\frac{n-1}{1-|z|^2} f_{n-1} \left( \frac{\sqrt{2}dz}{1-|z|^2} \right)^{\otimes n} \otimes d\bar{z} \\ &= -\frac{n-1}{\sqrt{2}} \varphi_{n-1} \otimes \frac{2dz \otimes d\bar{z}}{(1-|z|^2)^2} \\ &= -\frac{n-1}{\sqrt{2}} \varphi_{n-1} \otimes \omega\end{aligned}$$

for  $n \geq 1$ . Here we denoted

$$\omega := \frac{2dz \otimes d\bar{z}}{(1-|z|^2)^2} = g(z)dz \otimes d\bar{z},$$

a  $\Gamma$ -invariant  $(1, 1)$ -differential induced from the Poincaré metric.

#### 4. $L^2$ -ESTIMATE OF THE FORMAL SOLUTION

In this section, we shall compute the  $L^2$ -minimal solution to the system of the  $\bar{\partial}$ -equations on  $\Sigma$ ,

$$\bar{\partial}\varphi_0 = 0, \quad \bar{\partial}\varphi_n = -\frac{n-1}{\sqrt{2}} \varphi_{n-1} \otimes \omega \quad (n \geq 1),$$

where  $\varphi_n \in C^{(0,0)}(\Sigma, K_\Sigma^{\otimes n})$  and  $\omega = 2dz \otimes d\bar{z}/(1-|z|^2)^2$ .

**4.1. Multiplication by  $\omega$ .** We need some properties of the multiplication map

$$a_{n-1}: C^{(0,0)}(\Sigma, K_\Sigma^{\otimes(n-1)}) \rightarrow C^{(0,1)}(\Sigma, K_\Sigma^{\otimes n}), \quad u \mapsto u \otimes \omega$$

which appeared in the right hand side of our equation.

**Lemma 4.1.** *The map  $a_{n-1}$  is norm-preserving linear isomorphism, and it holds that*

$$a_{n-1} \left( \text{Ker}(\square_{n-1}^{(0)} - \lambda I) \right) = \text{Ker}(\square_n^{(1)} - (\lambda + n - 1)I).$$

*Proof.* It is clear that  $a_{n-1}$  is linear and bijective since  $\omega$  is non-vanishing. Let us see that  $a_{n-1}$  is norm-preserving. Take  $u \in C^{(0,0)}(\Sigma, K_\Sigma^{\otimes(n-1)})$ , then,

$$\|u \otimes \omega\|^2 = \int_\Sigma |u \otimes \omega|_g^2 \omega_g = \int_\Sigma |u|_g^2 |\omega|_g^2 \omega_g = \int_\Sigma |u|_g^2 \omega_g = \|u\|^2.$$

Note that our normalization implies

$$|\omega|_g^2 = \frac{4}{(1-|z|^2)^4} |dz|_g^2 = 1.$$

To show the correspondence of eigenforms, it suffices to prove the identity

$$\square_n^{(1)} \circ a_{n-1} = a_{n-1} \circ \square_{n-1}^{(0)} + (n-1)a_{n-1}.$$

Take  $u \in C^{(0,0)}(\Sigma, K_\Sigma^{\otimes(n-1)})$  and write  $u = u(z)(dz)^{\otimes(n-1)}$  in the uniformizing coordinate, then,

$$\begin{aligned}\square_n^{(1)}(u \otimes \omega) &= \frac{\partial}{\partial \bar{z}} \left( \frac{-1}{g} \left( \frac{\partial(gu)}{\partial z} + \frac{\partial \log g^{-n}}{\partial z} gu \right) \right) (dz)^{\otimes n} \otimes d\bar{z} \\ &= \frac{\partial}{\partial \bar{z}} \left( -\frac{\partial u}{\partial z} + (n-1) \frac{\partial \log g}{\partial z} u \right) (dz)^{\otimes n} \otimes d\bar{z} \\ &= \left( -\frac{\partial^2 u}{\partial z \partial \bar{z}} + (n-1) \frac{\partial \log g}{\partial z} \frac{\partial u}{\partial \bar{z}} + (n-1) gu \right) (dz)^{\otimes n} \otimes d\bar{z},\end{aligned}$$

where our normalization is

$$\begin{aligned}\frac{\partial \log g}{\partial z} &= -2 \frac{\partial \log(1-|z|^2)}{\partial z} = \frac{2\bar{z}}{(1-|z|^2)}, \\ \frac{\partial^2 \log g}{\partial z \partial \bar{z}} &= \frac{\partial}{\partial \bar{z}} \frac{2\bar{z}}{(1-|z|^2)} = \frac{2}{(1-|z|^2)^2} = g.\end{aligned}$$

On the other hand, in a coordinate trivializing  $K_\Sigma^{\otimes(n-1)}$ ,

$$\square_{n-1}^{(0)} u = \frac{-1}{g} \left( \frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{\partial \log g^{-(n-1)}}{\partial z} \frac{\partial u}{\partial \bar{z}} \right) (dz)^{\otimes(n-1)},$$

and this completes the proof.  $\square$

**4.2.  $L^2$ -minimal solution.** We shall solve the system of  $\bar{\partial}$ -equation inductively from a given holomorphic differential  $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$  by picking  $L^2$ -minimal solutions, and compute their  $L^2$  norms precisely.

**Proposition 4.2.** *Let  $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$  for some  $N \geq 1$ . Then, we have a solution  $\{\varphi_n\}_{n=0}^\infty$ ,  $\varphi_n \in C^{(0,0)}(\Sigma, K_\Sigma^{\otimes n})$ , to*

$$\bar{\partial}\varphi_0 = 0, \quad \bar{\partial}\varphi_n = -\frac{n-1}{\sqrt{2}}\varphi_{n-1} \otimes \omega \quad (n \geq 1)$$

that satisfies  $\varphi_n = 0$  ( $n < N$ ),  $\varphi_N = \psi$  and

$$\|\varphi_{N+m}\|^2 = \frac{(2N-1)!}{\{(N-1)!\}^2} \frac{\{(N+m-1)!\}^2}{m!(2N+m-1)!} \|\psi\|^2$$

for  $m \geq 1$ .

*Proof.* We put  $\varphi_n := 0$  ( $n < N$ ),  $\varphi_N := \psi$ , and

$$\varphi_{N+m} := \bar{\partial}_{N+m}^* G_{N+m}^{(1)} \left( -\frac{N+m-1}{\sqrt{2}} \varphi_{N+m-1} \otimes \omega \right)$$

for  $m \geq 1$  inductively. From Lemma 2.1,  $\varphi_{N+m}$  gives the  $L^2$ -minimal solution to

$$\bar{\partial}\varphi_{N+m} = -\frac{N+m-1}{\sqrt{2}} \varphi_{N+m-1} \otimes \omega.$$

We shall show

**Claim.** *Each  $\varphi_{N+m}$  is an eigenform of  $\square_{N+m}^{(0)}$  with eigenvalue  $E_{N,m}$ ,*

$$E_{N,m} := N + (N+1) + \cdots + (N+m-1) = \frac{m(2N+m-1)}{2}.$$

Let us prove this claim by induction. The first case  $m = 0$  is clear since  $\varphi_N = \psi$  is holomorphic, hence, of eigenvalue  $0 = E_{N,0}$ . Now assume the case  $m - 1$  and show it for  $m$ . The assumption and Lemma 4.1 yields that  $\varphi_{N+m-1} \otimes \omega$  is an eigenform of  $\square_{N+m}^{(1)}$  with eigenvalue

$$E_{N,m-1} + (N + m - 1) = E_{N,m}.$$

In particular,

$$G_{N+m}^{(1)}(\varphi_{N+m-1} \otimes \omega) = \frac{1}{E_{N,m}} \varphi_{N+m-1} \otimes \omega.$$

Therefore,

$$\begin{aligned} \square_{N+m}^{(0)} \varphi_{N+m} &= (\bar{\partial}_{N+m}^* \bar{\partial}) \bar{\partial}_{N+m}^* G_{N+m}^{(1)} \left( -\frac{N+m-1}{\sqrt{2}} \varphi_{N+m-1} \otimes \omega \right) \\ &= \bar{\partial}_{N+m}^* \square_{N+m}^{(1)} G_{N+m}^{(1)} \left( -\frac{N+m-1}{\sqrt{2}} \varphi_{N+m-1} \otimes \omega \right) \\ &= \bar{\partial}_{N+m}^* \left( -\frac{N+m-1}{\sqrt{2}} \varphi_{N+m-1} \otimes \omega \right) \\ &= \bar{\partial}_{N+m}^* \left( -\frac{N+m-1}{\sqrt{2}} E_{N,m} G_{N+m}^{(1)} \varphi_{N+m-1} \otimes \omega \right) \\ &= E_{N,m} \varphi_{N+m} \end{aligned}$$

and we finish the proof for this claim.

Now we shall complete the proof of Proposition 4.2. From the expression of  $\varphi_{N+m}$ , it follows that

$$\begin{aligned} \|\varphi_{N+m}\|^2 &= \left( \frac{N+m-1}{\sqrt{2}} \right)^2 \|\bar{\partial}_{N+m}^* G_{N+m}^{(1)}(\varphi_{N+m-1} \otimes \omega)\|^2 \\ &= \frac{(N+m-1)^2}{2} \langle G_{N+m}^{(1)}(\varphi_{N+m-1} \otimes \omega), \square_{N+m}^{(1)} G_{N+m}^{(1)}(\varphi_{N+m-1} \otimes \omega) \rangle \\ &= \frac{(N+m-1)^2}{2} \left\langle \frac{1}{E_{N,m}} \varphi_{N+m-1} \otimes \omega, \varphi_{N+m-1} \otimes \omega \right\rangle \\ &= \frac{(N+m-1)^2}{2E_{N,m}} \|\varphi_{N+m-1} \otimes \omega\|^2 = \frac{(N+m-1)^2}{2E_{N,m}} \|\varphi_{N+m-1}\|^2. \end{aligned}$$

This yields

$$\begin{aligned} \|\varphi_{N+m}\|^2 &= \frac{(N+m-1)^2}{2E_{N,m}} \frac{(N+m-2)^2}{2E_{N,m-1}} \|\varphi_{N+m-2}\|^2 \\ &= \left( \prod_{j=1}^m \frac{(N+m-j)^2}{2E_{N,m-j+1}} \right) \|\varphi_N\|^2 \\ &= \left( \prod_{j=1}^m \frac{(N+m-j)^2}{(m-j+1)(2N+m-j)} \right) \|\psi\|^2 \\ &= \frac{(2N-1)!}{\{(N-1)!\}^2} \frac{\{(N+m-1)!\}^2}{m!(2N+m-1)!} \|\psi\|^2 \end{aligned}$$

inductively, and the proof is completed.  $\square$

We shall denote the  $L^2$ -minimal solutions we have obtained by  $\widehat{I}(\psi) := \{\varphi_n\}_{n=0}^\infty$ , namely, we have constructed a map

$$\widehat{I}: \bigoplus_{N=0}^{\infty} H^0(\Sigma, K_{\Sigma}^{\otimes N}) \rightarrow \prod_{n=0}^{\infty} C^{0,0}(\Sigma, K_{\Sigma}^{\otimes n})$$

by letting

$$\widehat{I}(\psi) := \{\psi, 0, 0, \dots\}$$

for  $\psi \in H^0(\Sigma, K_{\Sigma}^{\otimes 0})$ , constant functions.

## 5. CONVERGENCE OF THE FORMAL SOLUTION

In this section, we shall prove that the formal holomorphic function whose Taylor coefficients along  $D$  are given by the solutions in Proposition 4.2 actually defines a genuine holomorphic function on  $\Omega$ , and in particular, it lives in the weighted Bergman space  $A_{\alpha}^2(\Omega)$  for all the  $\alpha > -1$ .

**5.1.  $L^2$  norms in terms of the Taylor coefficients.** Let  $f$  be a measurable function on  $\Omega$ , and we assume that  $f$  is holomorphic along all the fiber of  $\pi|_{\Omega}: \Omega \rightarrow \Sigma$ . Then, in the same manner as in §3.1, we can define the associated differentials  $\{\varphi_n\}$ , which are measurable differentials on  $\Sigma$ , by expanding  $f$  on each fiber of  $\pi|_{\Omega}$  at the intersection with the maximal compact analytic set  $D$ . We shall show the expression for the weighted  $L^2$  norm of  $f$  in terms of  $L^2$  norms of  $\varphi_n$ .

**Lemma 5.1.** *Let  $f$  be a measurable function on  $\Omega$  which is holomorphic along all the fiber, and denote by  $\{\varphi_n\}$  the associated differentials of  $f$  on  $\Sigma$ . Then, we have*

$$\|f\|_{\alpha}^2 = \pi \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+2+\alpha)} \|\varphi_n\|^2.$$

*Proof.* Denote by  $R \subset \mathbb{D}$  the fundamental domain of the universal covering map  $\mathbb{D} \rightarrow \Sigma$ . Then, the holomorphic disk bundle  $\pi|_{\Omega}: \Omega \rightarrow \Sigma$  is trivialized over  $R$  and it is enough to compute the  $L^2$  norm on  $\Omega' := \pi^{-1}(R) \cap \Omega \simeq R \times \mathbb{D}$ :

$$\begin{aligned} \Gamma(\alpha+1) \|f\|_{\alpha}^2 &= \int_{\Omega'} |f(z, w)|^2 \frac{4\delta^{\alpha}}{|1-\bar{z}w|^4} \frac{i}{2} dz \wedge d\bar{z} \wedge \frac{i}{2} dw \wedge d\bar{w} \\ &= \int_R 4d\lambda_z \int_{\mathbb{D}} |f(z, w)|^2 \frac{\delta^{\alpha}}{|1-\bar{z}w|^4} d\lambda_w \\ &= \int_R 4d\lambda_z \int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} f_n(z) \left( \frac{w-z}{1-\bar{z}w} \right)^n \right|^2 \frac{\delta^{\alpha}}{|1-\bar{z}w|^4} d\lambda_w \end{aligned}$$

where  $\lambda$  denotes the two-dimensional Lebesgue measure on  $\mathbb{D}$ . By changing the coordinate  $t = (w-z)/(1-\bar{z}w)$  on  $\mathbb{D}$ , since

$$d\lambda_t = \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} d\lambda_w,$$

we have

$$\begin{aligned}
\Gamma(\alpha + 1) \|f\|_\alpha^2 &= \int_R \frac{4d\lambda_z}{(1 - |z|^2)^2} \int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} f_n(z) t^n \right|^2 (1 - |t|^2)^\alpha d\lambda_t \\
&= \sum_{n=0}^{\infty} \int_R |f_n(z)|^2 \frac{4d\lambda_z}{(1 - |z|^2)^2} \int_{\mathbb{D}} |t|^{2n} (1 - |t|^2)^\alpha d\lambda_t \\
&= 2\pi \sum_{n=0}^{\infty} \int_{\Sigma} |\varphi_n|_g^2 \omega_g \int_0^1 r^{2n+1} (1 - r^2)^\alpha dr \\
&= \pi \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \|\varphi_n\|^2 \frac{\Gamma(n + 1)}{\Gamma(n + 2 + \alpha)},
\end{aligned}$$

and the proof is completed.  $\square$

**5.2. Convergence of the formal solution.** Let  $\{\varphi_n\} = \widehat{I}(\psi)$  the solution to the system of  $\bar{\partial}$ -equation obtained in Proposition 4.2 from a given holomorphic differential  $\psi \in H^0(\Sigma, K_{\Sigma}^{\otimes N})$ , and consider formal power series

$$f(z, w) := \sum_{n=0}^{\infty} f_n(z) t^n = \sum_{n=0}^{\infty} f_n(z) \left( \frac{w - z}{1 - \bar{z}w} \right)^n$$

where

$$f_n(z) := \varphi_n \left( \frac{\sqrt{2}dz}{1 - |z|^2} \right)^{-n}$$

in the uniformizing coordinate. We shall show that  $f \in L_\alpha^2(\Omega)$  for any  $\alpha > -1$ .

**Proposition 5.2.** *The sequence of partial sums  $\{F_n := \sum_{m=0}^n f_m(z) t^m\}_n$  is Cauchy in  $L_\alpha^2(\Omega)$  for any  $\alpha > -1$*

*Proof.* From Lemma 5.1 and Proposition 4.2, we may estimate, for any  $n \geq 0$ ,

$$\begin{aligned}
\|F_{N+n}\|_\alpha^2 &= \left\| \sum_{m=0}^n f_{N+m}(z) t^{N+m} \right\|_\alpha^2 = \pi \sum_{m=0}^n \|\varphi_{N+m}\|^2 \frac{\Gamma(N + m + 1)}{\Gamma(N + m + \alpha + 2)} \\
&= \pi \|\psi\|^2 \sum_{m=0}^n \frac{\Gamma(N + m + 1)}{\Gamma(N + m + \alpha + 2)} \frac{(2N - 1)!}{\{(N - 1)!\}^2} \frac{\{(N + m - 1)!\}^2}{(2N + m - 1)!} \frac{1}{m!}.
\end{aligned}$$

Hence, it is enough to show the convergence of the series

$$\sum_{m=0}^{\infty} \frac{\Gamma(N + m + 1)}{\Gamma(N + m + 2 + \alpha)} \frac{(2N - 1)!}{\{(N - 1)!\}^2} \frac{\{(N + m - 1)!\}^2}{(2N + m - 1)!} \frac{1}{m!},$$

which turns out to be a special value of the generalized hypergeometric function

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{\Gamma(N + m + 1)}{\Gamma(N + m + 2 + \alpha)} \frac{(2N - 1)!}{\{(N - 1)!\}^2} \frac{\{(N + m - 1)!\}^2}{(2N + m - 1)!} \frac{1}{m!} \\
&= \frac{\Gamma(N + 1)}{\Gamma(N + 2 + \alpha)} \sum_{m=0}^{\infty} \frac{(N + 1)_m}{(N + 2 + \alpha)_m} \frac{(N)_m (N)_m}{(2N)_m} \frac{1}{m!} \\
&= \frac{\Gamma(N + 1)}{\Gamma(N + 2 + \alpha)} {}_3F_2 \left( \begin{matrix} N + 1, N, N \\ 2N, N + 2 + \alpha \end{matrix}; 1 \right)
\end{aligned}$$

where  $(N)_m := N(N+1)\dots(N+m-1)$  and

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}$$

is the generalized hypergeometric function. It is well-known that  ${}_pF_q$  has finite value at  $z = 1$  if  $\sum_{i=1}^p a_i < \sum_{j=1}^q b_j$ , and in our case this condition corresponds to  $\alpha > -1$ , which we assumed.  $\square$

We shall show that this  $f$  is actually holomorphic. Note that each  $F_n$  is not holomorphic, and Proposition 5.2 does not automatically imply the holomorphicity of  $f$ .

**Proposition 5.3.** *The function  $f$  constructed above is holomorphic on  $\Omega$ .*

*Proof.* It is enough to show that  $f$  is holomorphic in the weak sense: there exists  $\alpha > -1$ , for any test function  $\phi \in C^{(0,1)}(\Omega)$  with compact support in  $\Omega$ ,  $\langle \langle f, \bar{\partial}^* \phi \rangle \rangle_\alpha = 0$ . We choose  $\alpha = 1$ . Since  $F_n \rightarrow f$  in  $L_{(2),1}^{(0,0)}(\Omega)$  and  $\bar{\partial} F_n \in L_{(2),1}^{(0,1)}(\Omega)$ , we have

$$\left| \langle \langle f, \bar{\partial}^* \phi \rangle \rangle_1 \right| = \lim_{n \rightarrow \infty} \left| \langle \langle F_n, \bar{\partial}^* \phi \rangle \rangle_1 \right| = \lim_{n \rightarrow \infty} \left| \langle \langle \bar{\partial} F_n, \phi \rangle \rangle_1 \right| \leq \lim_{n \rightarrow \infty} \|\bar{\partial} F_n\|_1 \|\phi\|_1.$$

Hence, it is enough to show that  $\|\bar{\partial} F_n\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ . Thanks to the equation that  $\{\varphi_n\}$  obeys, we have

$$\begin{aligned} \frac{\partial F_n}{\partial \bar{z}} &= \sum_{k=0}^n \left( \frac{\partial f_k}{\partial \bar{z}} \left( \frac{w-z}{1-\bar{z}w} \right)^k + \frac{kz f_k}{1-|z|^2} \left( \frac{w-z}{1-\bar{z}w} \right)^k + \frac{k f_k}{1-|z|^2} \left( \frac{w-z}{1-\bar{z}w} \right)^{k+1} \right) \\ &= \frac{n f_n}{1-|z|^2} \left( \frac{w-z}{1-\bar{z}w} \right)^{n+1}, \end{aligned}$$

and

$$\begin{aligned} \|\bar{\partial} F_n\|_1^2 &= \left\| \frac{n f_n}{1-|z|^2} \left( \frac{w-z}{1-\bar{z}w} \right)^{n+1} d\bar{z} \right\|_1^2 \\ &= n^2 \int_{\Omega'} \frac{|f_n(z)|^2}{(1-|z|^2)^2} \left| \frac{w-z}{1-\bar{z}w} \right|^{2(n+1)} |d\bar{z}|_g^2 \frac{4\delta}{|1-\bar{z}w|^4} \frac{i}{2} dz \wedge d\bar{z} \wedge \frac{i}{2} dw \wedge d\bar{w} \\ &= \frac{n^2}{2} \int_R |f_n(z)|^2 \frac{i}{2} dz \wedge d\bar{z} \int_{\mathbb{D}} \left| \frac{w-z}{1-\bar{z}w} \right|^{2(n+1)} \frac{4\delta}{|1-\bar{z}w|^4} d\lambda_w. \end{aligned}$$

By changing the coordinate  $t = (w-z)/(1-\bar{z}w)$  on  $\mathbb{D}$ ,

$$\begin{aligned} \|\bar{\partial} F_n\|_1^2 &= \frac{n^2}{2} \int_R |f_n(z)|^2 \frac{i}{2} dz \wedge d\bar{z} \int_{\mathbb{D}} |t|^{2(n+1)} \frac{4(1-|t|^2)}{(1-|z|^2)^2} d\lambda_t \\ &= \frac{n^2}{2} \int_R |\varphi_n|_g^2 \omega_g \int_{\mathbb{D}} |t|^{2(n+1)} (1-|t|^2) d\lambda_t \\ &= \frac{\pi n^2}{2} \|\varphi_n\|_2^2 \frac{\Gamma(n+3/2)}{\Gamma(n+7/2)} = \frac{\pi n^2}{2(n+3/2)(n+5/2)} \|\varphi_n\|^2. \end{aligned}$$

On the other hand, using Stirling's formula, we estimate

$$\begin{aligned} \|\varphi_{N+m}\|^2 &= \frac{(2N-1)!\|\psi\|^2}{\{(N-1)!\}^2} \frac{\{(N+m-1)!\}^2}{m!(2N+m-1)!} \\ &\approx \frac{\{\sqrt{2\pi(N+m-1)}(N+m-1)^{N+m-1}e^{-(N+m-1)}\}^2}{\sqrt{2\pi mm^m}e^{-m}\sqrt{2\pi(2N+m-1)}(2N+m-1)^{2N+m-1}e^{-(2N+m-1)}} \\ &= e^{\frac{(1+\frac{N-1}{m})^{2m+2N-1}}{(1+\frac{2N-1}{m})^{m+2N-0.5}} \frac{1}{m}} \approx e^{1+\frac{2}{N-1}-\frac{1}{2N-1}} \frac{1}{m} \end{aligned}$$

as  $m \rightarrow \infty$ , hence, we conclude

$$\|\bar{\partial}F_n\|_1^2 = \frac{\pi}{2(1+3/2n)(1+5/2n)} O\left(\frac{1}{n}\right) \rightarrow 0$$

as  $n \rightarrow \infty$ .  $\square$

## 6. PROOF OF MAIN THEOREM

Now we shall prove our Main Theorem. First we construct the desired extension operator

$$I: R(\Sigma) \rightarrow \bigcap_{\alpha > -1} A_\alpha^2(\Omega) \subset \mathcal{O}(\Omega),$$

where  $R(\Sigma) := \bigoplus_{N=0}^\infty H^0(\Sigma, K_\Sigma^{\otimes N})$  is the canonical ring of  $\Sigma$ , by summarizing our argument in previous sections.

*Construction of the extension operator.* For constant functions in  $H^0(\Sigma, K_\Sigma^{\otimes 0})$ , we just map it to the same constant. For each  $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$ ,  $N \geq 1$ , Proposition 4.2 yields the  $L^2$ -minimal solution  $\hat{I}(\psi) = \{\varphi_n\}$  to the system of  $\bar{\partial}$ -equation. We use  $\{\varphi_n\}$  as the Taylor coefficients of a formal function defined along  $D$ , namely, consider formal power series

$$I(\psi)(z, w) := \sum_{n=0}^\infty f_n(z) t^n = \sum_{n=0}^\infty f_n(z) \left( \frac{w-z}{1-\bar{z}w} \right)^n$$

where

$$f_n(z) := \varphi_n \left( \frac{\sqrt{2}dz}{1-|z|^2} \right)^{-n}$$

in the uniformizing coordinate  $z$ . Then, Proposition 5.2 and 5.3 guarantee that this formal function  $I(\psi)$  actually defines a holomorphic function of the weighted Bergman space of order  $> -1$ . We extend the map  $I$  on the direct sum  $\bigoplus_{N=0}^\infty H^0(\Sigma, K_\Sigma^{\otimes N})$   $\mathbb{C}$ -linearly, and obtain the extension operator  $I$ .  $\square$

We will need later

**Lemma 6.1.** *The operator  $I$  does not depend on the choice of uniformizing coordinate.*

*Proof.* Let  $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$ ,  $N \geq 1$ , and  $\hat{I}(\psi) = \{\varphi_n\}$ . We take another uniformizing coordinate  $z'$  of  $\Sigma$ , namely, take  $\gamma \in \text{Aut}(\mathbb{D})$  arbitrary and let  $z' = \gamma z$ , which express  $\Sigma$  as  $\mathbb{D}/\gamma\Gamma\gamma^{-1}$ ,  $z' \in \mathbb{D}$ . We shall show that  $I(\psi)$  does not depend on the choice of uniformizing coordinate  $z$  or  $z'$ .

If we use  $z'$  as the coordinate,  $I(\psi)$  is given by a  $\gamma\Gamma\gamma^{-1}$ -invariant holomorphic function on  $(z', w') \in \mathbb{D} \times \mathbb{D}$ ,

$$I'(\psi)(z', w') := \sum_{n=0}^{\infty} f'_n(z') \left( \frac{w' - z'}{1 - \overline{z'}w'} \right)^n$$

where

$$f'_n(z') := \varphi_n \left( \frac{\sqrt{2}dz'}{1 - |z'|^2} \right)^{-n}.$$

We shall compute  $I'(\psi)$  in  $(z, w)$ -coordinate. Since the identification between two coordinates is given by

$$(z', w') = (\gamma z, \gamma w), \quad \gamma z = \frac{\alpha z + \beta}{\overline{\beta}z + \overline{\alpha}}$$

where  $|\alpha|^2 - |\beta|^2 = 1$ , the coefficients transform as

$$f'_n(\gamma z) = f_n(z) \left( \frac{\beta \overline{z} + \alpha}{\overline{\beta}z + \overline{\alpha}} \right)^{-n}$$

by a computation similar to that in §3.2, hence, we have

$$I'(\psi)(\gamma z, \gamma w) = \sum_{n=0}^{\infty} f_n(z) \left( \frac{\beta \overline{z} + \alpha}{\overline{\beta}z + \overline{\alpha}} \right)^{-n} \left( \frac{\gamma w - \gamma z}{1 - \overline{\gamma z} \gamma w} \right)^n = I(\psi)(z, w).$$

□

Now we are going to prove our Main Theorem. Let us check

**Proposition 6.2.** *The operator  $I$  is injective.*

*Proof.* The injectivity of  $I$  on each summand of  $\bigoplus_{n=0}^{\infty} H^0(\Sigma, K_{\Sigma}^{\otimes n})$  is clear since for each  $\psi \in H^0(\Sigma, K_{\Sigma}^{\otimes n})$  the  $n$ -th jet of  $I(\psi)$ ,  $[I(\psi)] \in H^0(D, \mathcal{I}_D^n / \mathcal{I}_D^{n+1})$  is identified with  $\psi$  itself via the isomorphism  $H^0(D, \mathcal{I}_D^n / \mathcal{I}_D^{n+1}) \simeq H^0(\Sigma, K_{\Sigma}^{\otimes n})$ . We shall show that  $I(H^0(\Sigma, K_{\Sigma}^{\otimes n}))$  and  $I(H^0(\Sigma, K_{\Sigma}^{\otimes m}))$  are orthogonal in  $L_0^2(\Omega)$  if  $n \neq m$ , then the injectivity of  $I$  follows.

Suppose  $n_1 \neq n_2$  and take  $\psi_1 \in H^0(\Sigma, K_{\Sigma}^{\otimes n_1})$  and  $\psi_2 \in H^0(\Sigma, K_{\Sigma}^{\otimes n_2})$ . Denote  $\widehat{I}(\psi_1) = \{\varphi_{1,n}\}$  and  $\widehat{I}(\psi_2) = \{\varphi_{2,n}\}$ . From Claim in the proof of Proposition 4.2,  $\varphi_{1,n}$  and  $\varphi_{2,n}$  have different eigenvalue for  $\square_n^{(0)}$ , hence, they are orthogonal in  $L^2(\Sigma, K_{\Sigma}^{\otimes n})$ . The computation in Lemma 5.1 therefore yields the orthogonality

$$\langle \langle I(\psi_1), I(\psi_2) \rangle \rangle_0 = \pi \sum_{n=0}^{\infty} \frac{\langle \langle \varphi_{1,n}, \varphi_{2,n} \rangle \rangle}{n+1} = 0,$$

and we complete the proof. □

Next we would like to check

**Proposition 6.3.** *The operator  $I$  has dense image in  $\mathcal{O}(\Omega)$  with respect to the compact open topology.*

We need some additional notation. Consider  $\Omega_{\varepsilon} := \{[z, w] \in \Omega \mid \delta(z, w) > \varepsilon\}$  for  $\varepsilon \in (0, 1)$ , an exhaustion of  $\Omega$ , and denote by  $A_0^2(\Omega_{\varepsilon})$  the unweighted Bergman



space on  $\Omega_\varepsilon$  with respect to  $dV$ . From the computation in Lemma 5.1, we know that the  $L^2$  inner product of  $A_0^2(\Omega_\varepsilon)$  by  $\langle \langle \cdot, \cdot \rangle \rangle_{0,\varepsilon}$  is expressed as

$$\begin{aligned} \langle \langle f_1, f_2 \rangle \rangle_{0,\varepsilon} &= 2\pi \sum_{n=0}^{\infty} \langle \langle \varphi_{1,n}, \varphi_{2,n} \rangle \rangle \int_0^{\sqrt{1-\varepsilon}} r^{2n+1} dr \\ &= \pi \sum_{n=0}^{\infty} \langle \langle \varphi_{1,n}, \varphi_{2,n} \rangle \rangle \frac{(1-\varepsilon)^{n+1}}{n+1} \end{aligned}$$

where  $\{\varphi_{j,n}\}_n$  are the associated differentials of  $f_j$  as in §3. We will also use

$$\Pi_{n,\lambda}^{(j)} : L_{(2)}^{(0,j)}(\Sigma, K_\Sigma^{\otimes n}) \rightarrow \text{Ker}(\square_n^{(j)} - \lambda I), \quad j = 0, 1,$$

the orthogonal projection to the  $\lambda$ -eigenspace of the  $\bar{\partial}$ -Laplacian, which is finite dimensional thus forms a closed subspace thanks to the compactness of  $\Sigma$ .

*Proof of Proposition 6.3.* It is enough to show that  $I(R(\Sigma))$  is dense in  $A_0^2(\Omega_\varepsilon)$ . Then, our claim follows since any holomorphic function  $f \in \mathcal{O}(\Omega)$  belongs to  $A_0^2(\Omega_\varepsilon)$ , and Cauchy's estimate implies that  $f \in \mathcal{O}(\Omega)$  is uniformly approximated on  $\Omega_\varepsilon$  by functions belonging to  $I(R(\Sigma))$  for any  $\varepsilon \in (0, 1)$ .

Take  $f \in A_0^2(\Omega_\varepsilon)$  which is orthogonal to  $I(R(\Sigma))$ . We shall deduce a contradiction by assuming  $f \not\equiv 0$ . Denote by  $\{\varphi_n\}$  the associated differentials of  $f$ . Since  $I(R(\Sigma))$  contains constant functions,  $f$  is non-constant and there exists  $N$  such that  $\varphi_n \equiv 0$  for  $n < N$ , and  $\varphi_N \not\equiv 0$ , which is holomorphic since  $[f] \in H^0(D, \mathcal{I}_D^N / \mathcal{I}_D^{N+1})$  is identified with  $\varphi_N \in H^0(\Sigma, K_\Sigma^N)$ . Showing  $\hat{I}(\varphi_N) = \{\Pi_{n,E_{N,n-N}}^{(0)} \varphi_n\}$  deduces a contradiction since

$$\begin{aligned} 0 &= \langle \langle f, I(\varphi_N) \rangle \rangle_{0,\varepsilon} = \pi \sum_{m=0}^{\infty} \langle \langle \varphi_{N+m}, \Pi_{N+m,E_{N,m}}^{(0)} \varphi_{N+m} \rangle \rangle \frac{(1-\varepsilon)^{N+m+1}}{N+m+1} \\ &= \pi \sum_{m=0}^{\infty} \|\Pi_{N+m,E_{N,m}}^{(0)} \varphi_{N+m}\|^2 \frac{(1-\varepsilon)^{N+m+1}}{N+m+1} \\ &\geq \|\varphi_N\|^2 \frac{(1-\varepsilon)^{N+1}}{N+1} > 0. \end{aligned}$$

Now we shall show  $\hat{I}(\varphi_N) = \{\Pi_{n,E_{N,n-N}}^{(0)} \varphi_n\}$  by induction. It holds for  $n \leq N$ . Assume it for  $n < N+m$  and consider the case  $n = N+m$ . Since  $f - I(\varphi_N)$  is holomorphic, its associated differentials obey

$$\bar{\partial}(\varphi_{N+m} - \varphi'_{N+m}) = -\frac{N+m-1}{\sqrt{2}}(\varphi_{N+m-1} - \varphi'_{N+m-1}) \otimes \omega$$

where  $\hat{I}(\varphi_N) = \{\varphi'_n\}$ . Note that  $\varphi'_n \in \text{Ker}(\square_n^{(0)} - E_{N,n-N}I)$ . By applying the projector  $\Pi_{N+m,E_{N,m}}^{(1)}$ , it follows from Lemma 4.1 and the assumption that

$$\begin{aligned} &\bar{\partial}(\Pi_{N+m,E_{N,m}}^{(0)} \varphi_{N+m} - \varphi'_{N+m}) \\ &= \frac{N+m-1}{\sqrt{2}} \Pi_{N+m-1,E_{N,m-1}}^{(0)} (\varphi_{N+m-1} - \varphi'_{N+m-1}) \otimes \omega = 0 \end{aligned}$$

Since  $\text{Ker } \bar{\partial} \perp \text{Ker}(\square_{N+m}^{(0)} - E_{N,m}I)$ , we conclude  $\varphi'_{N+m} = \Pi_{N+m,E_{N,m}}^{(0)} \varphi_{N+m}$ .  $\square$

The remaining thing to be shown is the expression of  $I$  given in the statement of Main Theorem. Let us see that the expression, which we temporarily denote by  $J(\psi)$ , gives a  $\Gamma$ -invariant holomorphic function on  $\mathbb{D} \times \mathbb{D}$ .

**Lemma 6.4.** *Let  $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$ ,  $N \geq 1$ , and write  $\psi = \psi(\tau)(d\tau)^{\otimes N}$  on the uniformizing coordinate  $z$ . Then,*

$$J(\psi)(z, w) := \frac{1}{B(N, N)} \int_z^w \left( \frac{(w - \tau)(\tau - z)}{w - z} \right)^{N-1} \psi(\tau) d\tau.$$

*defines a  $\Gamma$ -invariant holomorphic function on  $\mathbb{D} \times \mathbb{D}$ , and it is independent of the choice of uniformizing coordinate.*

*Proof.* Notice that

$$[w, \tau, z] := \frac{(w - z)d\tau}{(w - \tau)(\tau - z)} = \left( \frac{1}{w - \tau} + \frac{1}{\tau - z} \right) d\tau$$

is a meromorphic 1-form in  $(z, w, \tau) \in \mathbb{D} \times \mathbb{D} \times \mathbb{D}$  and invariant under the simultaneous action of  $\text{Aut}(\mathbb{D})$  on  $\mathbb{D} \times \mathbb{D} \times \mathbb{D}$  since it is a degenerate form of the cross ratio. Hence, when we fix distinct  $z, w \in \mathbb{D}$ , the integrand

$$\frac{1}{B(N, N)} \left( \frac{(w - \tau)(\tau - z)}{(w - z)d\tau} \right)^{\otimes(N-1)} \otimes \psi(\tau)(d\tau)^{\otimes N}$$

is a holomorphic 1-form in  $\tau$ , and the value  $J(\psi)(z, w)$  does not depend on the choice of integral path from  $z$  to  $w$ . The expression clearly shows that  $J(\psi)(z, w)$  is holomorphic in on  $\mathbb{D} \times \mathbb{D} \setminus \Delta$  and the invariance under  $\Gamma$  follows from

$$\begin{aligned} (\gamma^* J(\psi))(z, w) &= J(\psi)(\gamma(z), \gamma(w)) \\ &= \int_{\gamma(z)}^{\gamma(w)} \frac{1}{B(N, N)} \left( \frac{(\gamma(w) - \tau)(\tau - \gamma(z))}{(\gamma(w) - \gamma(z))d\tau} \right)^{\otimes(N-1)} \otimes \psi(\tau)(d\tau)^{\otimes N} \\ &= \int_z^w \frac{1}{B(N, N)} \left( \frac{(\gamma(w) - \gamma(\tau'))(\gamma(\tau') - \gamma(z))}{(\gamma(w) - \gamma(z))d\gamma(\tau')} \right)^{\otimes(N-1)} \otimes \psi(\gamma(\tau'))(d\gamma(\tau'))^{\otimes N} \\ &= \int_z^w \frac{1}{B(N, N)} \left( \frac{(w - \tau')(\tau' - z)}{(w - z)d\tau'} \right)^{\otimes(N-1)} \otimes \gamma^* (\psi(\tau)(d\tau)^{\otimes N}) \\ &= J(\gamma^* \psi)(z, w) = J(\psi)(z, w), \end{aligned}$$

thanks to  $\Gamma$ -invariance of  $\psi(\tau)(d\tau)^{\otimes N}$  and  $\text{Aut}(\mathbb{D})$ -invariance of  $[w, \tau, z]$ , where  $\gamma \in \Gamma$ , and we change the variable of the integral by  $\tau = \gamma(\tau')$ . Along  $\Delta$ ,  $J(\psi)$  behaves as follows:

$$\begin{aligned} J(\psi)(z, w) &= \int_z^{z+(w-z)} \frac{1}{B(N, N)} \left( \frac{(z + (w - z) - \tau)(\tau - z)}{(w - z)} \right)^{N-1} \psi(\tau) d\tau \\ &= \int_0^1 \frac{1}{B(N, N)} \left( \frac{((w - z) - (w - z)s)(w - z)s}{(w - z)} \right)^{N-1} \psi(z + (w - z)s)(w - z) ds \\ &= (w - z)^N \int_0^1 \frac{1}{B(N, N)} ((1 - s)s)^{N-1} \psi(z + (w - z)s) ds \\ &= (w - z)^N \int_0^1 \psi(z + (w - z)s) \beta_N(ds) \end{aligned}$$

where we chosen the integral path as  $\tau = z + s(w - z)$ ,  $0 \leq s \leq 1$ , and

$$\beta_N(dt) := \frac{s^{N-1}(1-s)^{N-1}ds}{B(N, N)} = \frac{(2N-1)!}{(N-1)!(N-1)!} s^{N-1}(1-s)^{N-1}ds$$

denotes the beta distribution with parameters  $(N, N)$  on  $s \in [0, 1]$ . Hence,  $J(\psi)$  has zero of order  $N$  along  $\Delta$ , and is a  $\Gamma$ -invariant holomorphic function on  $\mathbb{D} \times \mathbb{D}$ .

We take another uniformizing coordinate  $z'$  of  $\Sigma$  given by  $z' = \gamma z$  for arbitrary  $\gamma \in \text{Aut}(\mathbb{D})$ , and denote by  $J'(\psi)$  the path integral computed using  $z'$ -coordinate. Then, the given differential  $\psi$  is expressed by pull-back  $(\gamma^{-1})^*\psi \in H^0(\mathbb{D}, (K_{\mathbb{D}})^{\otimes N})$  in  $z'$ -coordinate, hence, the same computation as in the proof for  $\Gamma$ -invariance of  $J(\psi)$  yields

$$J(\psi)(\gamma^{-1}z, \gamma^{-1}w) = J'(\psi)(z, w),$$

which means that  $J(\psi)$  does not depend on the choice of uniformizing coordinate  $z$  or  $z'$ .  $\square$

Now we are going to show that  $J$  actually gives an expression for  $I$ .

**Proposition 6.5.** *For any  $\psi \in H^0(\Sigma, K_{\Sigma}^{\otimes N})$ ,  $N \geq 1$ ,  $I(\psi) = J(\psi)$  holds.*

*Proof.* It is enough to show it on  $\{0\} \times \mathbb{D}$ , namely,

$$I(\psi)(0, w) = J(\psi)(0, w) = w^N \int_0^1 \psi(sw) \beta_N(ds)$$

for each  $w \in \mathbb{D}$  since all our argument does not depend on the choice of uniformizing coordinate  $z$ . More precisely, to show

$$I(\psi)(z_0, w_0) = J(\psi)(z_0, w_0)$$

at  $(z_0, w_0) \in \mathbb{D} \times \mathbb{D}$ , we take  $\gamma_0 \in \text{Aut}(\mathbb{D})$  such that  $\gamma_0(0) = z_0$ , and replace the original uniformization  $\mathbb{D}/\Gamma$  by another uniformization  $\mathbb{D}/\gamma_0^{-1}\Gamma\gamma_0$ . Lemma 6.1 and 6.4 yield

$$\begin{aligned} I(\psi)(z_0, w_0) &= I'(\psi)(0, \gamma_0^{-1}(w_0)), \\ J(\psi)(z_0, w_0) &= J'(\psi)(0, \gamma_0^{-1}(w_0)) \end{aligned}$$

respectively, where  $I'$  and  $J'$  denote the operators computed in the new uniformizing coordinate. Hence, showing  $I(\gamma_0^*\psi)(0, \gamma_0^{-1}(w_0)) = J(\gamma_0^*\psi)(0, \gamma_0^{-1}(w_0))$  is enough to conclude.

We shall write down  $I(\psi)(0, w)$  explicitly. The  $L^2$ -minimal solution  $\hat{I}(\psi) = \{\varphi_n\}$  satisfies  $\varphi_n = 0$  ( $n < N$ ),  $\varphi_N = \psi$ , and

$$\begin{aligned} \varphi_{N+m} &= -\frac{N+m-1}{\sqrt{2}E_{N,m}} \bar{\partial}_{N+m}^* (\varphi_{N+m-1} \otimes \omega) \\ &= -\frac{\sqrt{2}(N+m-1)}{m(2N+m-1)} \bar{\partial}_{N+m}^* (\varphi_{N+m-1} \otimes \omega). \end{aligned}$$

We write

$$\varphi_{N+m-1} = f_{N+m-1} \left( \frac{\sqrt{2}dz}{1-|z|^2} \right)^{\otimes(N+m-1)} = f_{N+m-1} \sqrt{g}^{N+m-1} (dz)^{\otimes(N+m-1)}.$$

Then,

$$\begin{aligned}
& \overline{\partial}_{N+m}^* (\varphi_{N+m-1} \otimes \omega) \\
&= \frac{-1}{g g^{-(N+m)}} \frac{\partial(g^{-(N+m)} f_{N+m-1} \sqrt{g}^{N+m-1} g)}{\partial z} (dz)^{\otimes(N+m)} \\
&= -g^{N+m-1} \frac{\partial(\sqrt{g}^{-(N+m-1)} f_{N+m-1})}{\partial z} (dz)^{\otimes(N+m)} \\
&= -\sqrt{g}^{N+m-2} \frac{\partial(\sqrt{g}^{-(N+m-1)} f_{N+m-1})}{\partial z} (\sqrt{g} dz)^{\otimes(N+m)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
f_{N+m}(z) &= \frac{\sqrt{2}(N+m-1)}{m(2N+m-1)} \sqrt{g}^{N+m-2} \frac{\partial(\sqrt{g}^{-(N+m-1)} f_{N+m-1})}{\partial z} \\
&= \frac{\sqrt{2}(N+m-1)}{m(2N+m-1)} \frac{\sqrt{2}(N+m-2)}{(m-1)(2N+m-2)} \sqrt{g}^{N+m-2} \frac{\partial}{\partial z} \frac{1}{g} \frac{\partial(\sqrt{g}^{-(N+m-2)} f_{N+m-2})}{\partial z} \\
&= \frac{(2N-1)!}{(N-1)!} \frac{\sqrt{2}^m (N+m-1)!}{m!(2N+m-1)!} \sqrt{g}^{N+m-2} \left( \frac{\partial}{\partial z} \frac{1}{g} \right)^{\circ(m-1)} \frac{\partial \sqrt{g}^{-N} f_N}{\partial z}.
\end{aligned}$$

Using explicit expression of  $g$ ,

$$\begin{aligned}
& f_{N+m}(z) \\
&= \frac{(2N-1)!}{(N-1)!} \frac{(N+m-1)!}{m!(2N+m-1)!} \frac{1}{(1-|z|^2)^{N+m-2}} \left( \frac{\partial}{\partial z} (1-|z|^2)^2 \right)^{\circ(m-1)} \frac{\partial}{\partial z} (1-|z|^2)^N f_N
\end{aligned}$$

follows, and at  $z = 0$ ,

$$f_{N+m}(0) = \frac{(2N-1)!}{(N-1)!} \frac{(N+m-1)!}{m!(2N+m-1)!} \frac{\partial^m f_N}{\partial z^m}(0).$$

Note that when a  $z$ -derivative hit at  $(1-|z|^2)$ , a factor  $\bar{z}$  appears and the terms keeping it must vanish at  $z = 0$ .

We therefore showed that the holomorphic function  $I(\psi)$  has the expansion

$$I(\psi)(0, w) = \frac{(2N-1)!}{(N-1)!} \sum_{m=0}^{\infty} \frac{(N+m-1)!}{(2N+m-1)!} \frac{1}{m!} \frac{\partial^m \psi}{\partial z^m}(0) w^{N+m}.$$

Hence, it satisfies the differential equation

$$\begin{aligned}
\frac{\partial^N}{\partial w^N} (w^{N-1} I(\psi)(0, w)) &= \frac{(2N-1)!}{(N-1)!} \frac{\partial^N}{\partial w^N} \sum_{m=0}^{\infty} \frac{(N+m-1)!}{(2N+m-1)!} \frac{1}{m!} \frac{\partial^m \psi}{\partial z^m}(0) w^{2N+m-1} \\
&= \frac{(2N-1)!}{(N-1)!} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \psi}{\partial z^m}(0) w^{N+m-1} \\
&= \frac{(2N-1)!}{(N-1)!} w^{N-1} \psi(w).
\end{aligned}$$

Using an iterated integral on a path from 0 to  $w$ , in particular, taking the path as the segment joining 0 and  $w$ , we obtain

$$\begin{aligned}
I(\psi)(0, w) &= \frac{(2N-1)!}{(N-1)!} \frac{1}{w^{N-1}} \int_0^w d\sigma_N \cdots \int_0^{\sigma_3} d\sigma_2 \int_0^{\sigma_2} \sigma_1^{N-1} \psi(\sigma_1) d\sigma_1 \\
&= \frac{(2N-1)!}{(N-1)!} \frac{1}{w^{N-1}} \int_0^1 w ds_N \cdots \int_0^{s_3} w ds_2 \int_0^{s_2} (s_1 w)^{N-1} \psi(ws_1) w ds_1 \\
&= \frac{(2N-1)!}{(N-1)!} w^N \int_{\{0 < s_1 < \cdots < s_N < 1\}} s_1^{N-1} \psi(s_1 w) ds_1 ds_2 \cdots ds_N \\
&= \frac{(2N-1)!}{(N-1)!} w^N \int_0^1 ds_1 s_1^{N-1} \psi(s_1 w) \int_{s_1}^1 ds_2 \cdots \int_{s_{N-1}}^1 ds_N \\
&= \frac{(2N-1)!}{(N-1)!} w^N \int_0^1 \frac{s_1^{N-1} (1-s_1)^{N-1}}{(N-1)!} \psi(s_1 w) ds_1 \\
&= w^N \int_0^1 \psi(sw) \beta_N(ds).
\end{aligned}$$

We therefore have  $I(\psi)(0, w) = J(\psi)(0, w)$ , and finished the proof for Main Theorem.  $\square$

## 7. APPLICATIONS

We shall give two applications of our description of  $\mathcal{O}(\Omega)$ .

**7.1. The vanishing of Hardy space.** As we have shown, the intersection of weighted Bergman spaces  $\bigcap_{\alpha > -1} A_\alpha^2(\Omega)$  is infinite dimensional. However,

**Corollary 2.** *The Hardy space  $A_{-1}^2(\Omega)$  consists only of constant functions.*

The Hardy space of  $\Omega$  is defined by

$$A_{-1}^2(\Omega) := \{f \in \mathcal{O}(\Omega) \mid \|f\|_{-1} < \infty\}.$$

Here the Hardy norm  $\|\cdot\|_{-1}$  is defined by

$$\|f\|_{-1}^2 := \pi \sum_{n=0}^{\infty} \|\varphi_n\|^2.$$

where  $\{\varphi_n\}$  are the associated differentials of  $f$ .

*Proof.* Let  $f \in \mathcal{O}(\Omega)$ . We denote the associated differentials of  $f$  by  $\{\varphi_n\}$ . If  $f$  is non-constant, we find  $N$  such that  $\varphi_N \not\equiv 0$ , and  $\varphi_n \equiv 0$  for  $n < N$ . From the argument in §3.2,  $\psi := \varphi_N$  is a holomorphic  $N$ -differential on  $\Sigma$ . Then, repeating the computation in the proof of Proposition 5.3 with the orthogonal projections

$$\Pi_{n,\lambda}^{(0)} : L_{(2)}^{(0,0)}(\Sigma, K_\Sigma^{\otimes n}) \rightarrow \text{Ker}(\square_n^{(0)} - \lambda I),$$

we have

$$\begin{aligned}
\|\Pi_{N+m, E_{N,m}}^{(0)} \varphi_{N+m}\|^2 &= \frac{(N+m-1)^2}{2E_{N,m}} \|\Pi_{N+m-1, E_{N,m-1}}^{(0)} \varphi_{N+m-1}\|^2 \\
&= \frac{(2N-1)!}{\{(N-1)!\}^2} \frac{\{(N+m-1)!\}^2}{m!(2N+m-1)!} \|\Pi_{N,0}^{(0)} \varphi_N\|^2
\end{aligned}$$

for  $m \geq 1$ . Since  $\|\Pi_{N,0}^{(0)}\varphi_N\| = \|\varphi_N\| \neq 0$ , it follows from the proof of Proposition 5.2 that

$$\|\Pi_{N+m,E_{N,m}}^{(0)}\varphi_{N+m}\|^2 \approx C \frac{1}{m}$$

as  $m \rightarrow \infty$  for some constant  $C > 0$ . Hence,

$$\|f\|_{-1}^2 = \sum_{n=0}^{\infty} \|\varphi_n\|^2 = \sum_{m=0}^{\infty} \|\varphi_{N+m}\|^2 \geq \sum_{m=0}^{\infty} \|\Pi_{N+m,E_{N,m}}^{(0)}\varphi_{N+m}\|^2$$

cannot converge.  $\square$

**7.2. Forelli–Rudin construction for  $\Omega$ .** Let  $\alpha > -1$ . Since we have constructed a map

$$I: R(\Sigma) = \bigoplus_{N=0}^{\infty} H^0(\Sigma, K_{\Sigma}^{\otimes N}) \hookrightarrow A_{\alpha}^2(\Omega)$$

with dense image, we are able to express the weighted Bergman kernel of  $A_{\alpha}^2(\Omega)$  in terms of the Bergman kernels of  $H^0(\Sigma, K_{\Sigma}^{\otimes N})$ .

**Corollary 3.** *For  $\alpha > -1$ , the weighted Bergman kernel  $B_{\alpha}((z, w); (z', w'))$  of  $A_{\alpha}^2(\Omega)$  has the following expression*

$$\begin{aligned} & B_{\alpha}((z, w); (z', w')) \\ &= \frac{\Gamma(\alpha+2)}{\pi^2(4g-4)} + \frac{1}{\pi} \sum_{N=1}^{\infty} \frac{1}{c_{N,\alpha}} \frac{1}{B(N, N)^2} \int_{\tau \in \overline{zw}} \int_{\tau' \in \overline{z'w'}} \frac{B_N(\tau, \tau')(d\tau \otimes \overline{d\tau'})^{\otimes N}}{([w, \tau, z] \otimes [\overline{w'}, \overline{\tau'}, \overline{z'}])^{\otimes(N-1)}} \end{aligned}$$

where  $B_N(\tau, \tau')(d\tau \otimes \overline{d\tau'})^{\otimes N}$  is the Bergman kernel of  $K_{\Sigma}^{\otimes N}$ ,

$$c_{N,\alpha} := \frac{\Gamma(N+1)}{\Gamma(N+2+\alpha)} {}_3F_2 \left( \begin{matrix} N+1, N, N \\ 2N, N+2+\alpha \end{matrix}; 1 \right),$$

and  $g$  is the genus of  $\Sigma$ .

*Proof.* We write down a complete orthonormal basis of  $A_{\alpha}^2(\Omega)$  using the map  $I$ . Pick an orthonormal basis  $\{\psi_{N,j} = \psi_{N,j}(\tau)(d\tau)^{\otimes N}\}_{j=1}^{d_N}$  for each  $H^0(\Sigma, K_{\Sigma}^{\otimes N})$ . Note that  $d_0 = 1$  and  $\psi_{0,1} = 1/\sqrt{\text{Vol}(\Sigma)} = 1/\sqrt{\pi(4g-4)}$  from the Gauss–Bonnet theorem. Recall that the Bergman kernel  $B_N$  of  $K_{\Sigma}^{\otimes N}$  is given by

$$B_N = B_N(\tau, \tau')(d\tau \otimes \overline{d\tau'})^{\otimes N} = \sum_{j=1}^{d_N} \psi_{N,j}(\tau)(d\tau)^{\otimes N} \otimes \overline{\psi_{N,j}(\tau')(d\tau')^{\otimes N}}.$$

Now we collect all the holomorphic functions  $\{I(\psi_{N,j})\}_{N,j}$ . It is clear in view of our discussion in §6 that this is a complete orthogonal basis of  $A_{\alpha}^2(\Omega)$ . The computation in the proof of Proposition 5.2 yields

$$\|I(\psi)\|_{\alpha}^2 = \pi \|\psi\|^2 \frac{\Gamma(N+1)}{\Gamma(N+2+\alpha)} {}_3F_2 \left( \begin{matrix} N+1, N, N \\ 2N, N+2+\alpha \end{matrix}; 1 \right)$$

for  $\psi \in H^0(\Sigma, K_{\Sigma}^{\otimes N})$ , hence,  $\{\sqrt{\Gamma(\alpha+2)}/\pi\sqrt{4g-4}\} \cup \{I(\psi_{N,j})/\sqrt{\pi c_{N,\alpha}}\}_{N \geq 1, j}$  is a complete orthonormal basis of  $A_{\alpha}^2(\Omega)$ . We therefore obtain the expression for the

Bergman kernel

$$\begin{aligned}
& B_\alpha((z, w); (z', w')) - \frac{\Gamma(\alpha + 2)}{\pi^2(4g - 4)} \\
&= \sum_{N=1}^{\infty} \sum_{j=1}^{d_N} \frac{1}{\pi c_{N,\alpha}} I(\psi_{N,j})(z, w) \overline{I(\psi_{N,j})(z', w')} \\
&= \frac{1}{\pi} \sum_{N=1}^{\infty} \frac{1}{c_{N,\alpha}} \sum_{j=1}^{d_N} \frac{1}{B(N, N)^2} \int_{\tau \in \overline{zw}} \frac{\psi_{N,j}(\tau)(d\tau)^{\otimes N}}{[w, \tau, z]^{\otimes(N-1)}} \overline{\int_{\tau' \in \overline{z'w'}} \frac{\psi_{N,j}(\tau')(d\tau')^{\otimes N}}{[w', \tau', z']^{\otimes(N-1)}}} \\
&= \frac{1}{\pi} \sum_{N=1}^{\infty} \frac{1}{c_{N,\alpha}} \frac{1}{B(N, N)^2} \int_{\tau \in \overline{zw}} \int_{\tau' \in \overline{z'w'}} \frac{B_N(\tau, \tau')(d\tau \otimes \overline{d\tau'})^{\otimes N}}{([w, \tau, z] \otimes [\overline{w'}, \tau', z'])^{\otimes(N-1)}}.
\end{aligned}$$

□

**7.3. Invariant holomorphic functions given by Poincaré series.** We conclude this paper by expressing the invariant holomorphic functions constructed by Ohsawa [21] using our integral operator  $I$ . Note that for  $N \geq 2$ ,  $\sum_{\gamma \in \Gamma} \gamma^* d\tau^{\otimes N} \in H^0(\mathbb{D}, K_{\mathbb{D}}^{\otimes N})$  gives a holomorphic differential  $\in H^0(\Sigma, K_{\Sigma}^{\otimes N})$ , and the convergence of Poincaré series is uniform on each compact  $\subset \mathbb{D}$ .

**Corollary 4.** *For  $N \geq 2$ , we have*

$$I\left(\sum_{\gamma \in \Gamma} \gamma^* d\tau^{\otimes N}\right)(z, w) = \sum_{\gamma \in \Gamma} (\gamma(z) - \gamma(w))^N.$$

*Proof.* By direct computation. We compute

$$\begin{aligned}
I\left(\sum_{\gamma \in \Gamma} \gamma^* d\tau^{\otimes N}\right)(z, w) &= \frac{1}{B(N, N)} \int_z^w \sum_{\gamma \in \Gamma} \frac{\gamma'(\tau)^N (w - \tau)^{N-1} (\tau - z)^{N-1}}{(w - z)^{N-1}} d\tau \\
&= \sum_{\gamma \in \Gamma} \frac{1}{B(N, N)} \int_{\gamma(z)}^{\gamma(w)} \frac{(\gamma(w) - \tau')^{N-1} (\tau' - \gamma(z))^{N-1}}{(\gamma(w) - \gamma(z))^{N-1}} d\tau',
\end{aligned}$$

where we introduced new coordinate  $\tau' = \gamma(\tau)$  on  $\mathbb{D}$ . Notice the invariance of  $[w, \tau, z]$ . On the other hand, we compute each summand using the segment  $\tau'(t) = \gamma(z) + t(\gamma(w) - \gamma(z))$ ,  $t \in [0, 1]$ ,

$$\begin{aligned}
& \frac{1}{B(N, N)} \int_{\gamma(z)}^{\gamma(w)} \frac{(\gamma(w) - \tau')^{N-1} (\tau' - \gamma(z))^{N-1}}{(\gamma(w) - \gamma(z))^{N-1}} d\tau' \\
&= (\gamma(w) - \gamma(z))^N \int_0^1 \frac{(1-t)^{N-1} t^{N-1} dt}{B(N, N)} \\
&= (\gamma(w) - \gamma(z))^N,
\end{aligned}$$

and this completes the proof. □

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